Bethe ansatz solution of zero-range process with nonuniform stationary state

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The eigenfunctions and eigenvalues of the master equation for zero-range process with totally asymmetric dynamics on a ring are found exactly using the Bethe ansatz weighted with the stationary weights of particle configurations. The Bethe ansatz applicability requires the rates of hopping of particles out of a site to be the q numbers $[n]_q$. This is a generalization of the rates of hopping of noninteracting particles equal to the occupation number n of a site of departure. The noninteracting case can be restored in the limit $q \to 1$. The limiting cases of the model for $q=0,\infty$ correspond to the totally asymmetric exclusion process, and the drop-push model, respectively. We analyze the partition function of the model and apply the Bethe ansatz to evaluate the generating function of the total distance traveled by particles at large time in the scaling limit. In case of nonzero interaction, $q \neq 1$, the generating function has the universal scaling form specific for the Kardar-Parizi-Zhang universality class.

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I. INTRODUCTION

The Bethe ansatz [1] is one of the most powerful tools to get exact results for the systems with many interacting degrees of freedom in low dimensions. The exact solutions of one-dimensional quantum spin chains and two-dimensional vertex models are classical examples of its application [2]. In the last decade, the Bethe ansatz was shown to be useful to study one-dimensional stochastic processes [3,4]. The first and most explored example is the asymmetric simple exclusion process (ASEP), which serves as a testing ground for many concepts of the nonequilibrium statistical physics [5]. Yet several other Bethe ansatz solvable models of nonequilibrium processes have been proposed such as the asymmetric diffusion models [6,7], generalizations of the drop-push model [8–10], and the asymmetric avalanche process (ASAP) [11].

Most of the models studied by the coordinate Bethe ansatz have a common property. That is, a system evolves to the stationary state, where all the particle configurations occur with the same probability. This property can be easily understood from the structure of the Bethe eigenfunction. Indeed, the stationary state is given by the ground state of evolution operator, which is the eigenfunction with zero eigenvalue and momentum. Such Bethe function does not depend on particle configuration at all and results in the equiprobable ensemble. Except for a few successful attempts to apply the Bethe ansatz to systems with nonuniform stationary state, such as the ASEP with blockage [12] or defect particle [13], there is not much progress in this direction.

On the other hand many interesting physical phenomena such as condensation in nonequilibrium systems [14], boundary induced phase transitions [15,16], or intermittent-continuous flow transition [17] become apparent from nontrivial form of the stationary state, which changes dramatically from one point of phase space to another. Typical example is the zero-range process (ZRP) served as a prototype of a one-dimensional nonequilibrium system exhibiting the condensation transition [14]. While its stationary measure has been studied in detail [18], the full dynamical description is still absent.

The aim of this paper is to obtain the Bethe ansatz solution of the ZRP. The paper is organized as follows. In Sec. II we use the Bethe ansatz to solve the eigenfunction and eigenvalue problem for the master equation of the ZRP and show that its applicability requires the rates of hopping of particles out of a site to be the q numbers. We show that with this choice of the rates the model is equivalent to the q-boson totally asymmetric diffusion model. In Sec. III we obtain the partition function of the ZRP with the rates obtained and evaluate some stationary correlations. In Sec. IV we apply the equations obtained from the Bethe ansatz solution to derive the expression for the generating function of the total distance traveled by particles in the large time limit. We summarize the results in Sec. V.

II. MASTER EQUATION OF ZERO-RANGE PROCESS

Let us consider the system of p particles on a ring consisting of N sites. Every site can hold an integer number of particles. Every moment of time, one particle can leave any occupied site, hopping to the next site clockwise. The rate of hopping $u(n_i)$ depends only on the occupation number n_i of the site of departure i. The stationary measure of such a process has been found to be a product measure [14], i.e., the probability of configuration $\{n_i\}$, specified by the occupation numbers $\{n_1,n_2,\ldots,n_N\}$, is, up to the normalization factor, given by the weight

$$W(\{n_i\}) = \prod_{i=1}^{N} f(n_i), \tag{1}$$

where one-site weight $f(n)$ is

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$$f(n) = \prod_{m=1}^{n} \frac{1}{u(m)} \tag{2}$$

if $n>0$, and $f(0)=1$.

Let us consider the probability $P_t(n_1,\ldots,n_N)$ for N sites to have occupation numbers $n_1,\ldots,n_N$ at time t. It obeys the master equation defined by the dynamics described:

$$\partial_t P_t(n_1,\ldots,n_N) = \sum_{\substack{k=1\\ n_k\neq 0}}^{N} [u(n_{k-1}+1)P_t(\ldots,n_{k-1}+1,n_k-1,\ldots) - u(n_k)P_t(n_1,\ldots,n_N)]. \tag{3}$$

Here, cyclic boundary condition, $n_{-1}\equiv n_N$, is imposed.

Another way to specify the configuration of system is to use the set of coordinates of p particles $\{x_i\}=\{x_1,\ldots,x_p\}$, rather than occupation numbers $\{n_1,\ldots,n_N\}$, the two ways being completely equivalent. While the explicit form of the master equation in such notations is more complicated, it turns out more appropriate for an analytic solution. The main idea of the solution is to use the Bethe ansatz for the function $P_t^0(x_1,\ldots,x_p)$ related with the solution of the master equation $P_t(x_1,\ldots,x_p)$ as follows:

$$P_t(x_1,\ldots,x_p) = W(\{n_i\})P_t^0(x_1,\ldots,x_p). \tag{4}$$

A. Two-particle sector

To explain the details we first consider the case $p=2$ subsequently generalizing it to the case of arbitrary p. Without loss of generality we can define

$$u(1)\equiv 1, \quad u(2)\equiv u>0. \tag{5}$$

Now, we are going to show that the solution, $P_t^0(x_1,x_2)$, of the master equation for noninteracting particles,

$$\partial_t P_t^0(x_1,x_2) = P_t^0(x_1-1,x_2) + P_t^0(x_1,x_2-1) - 2P_t^0(x_1,x_2), \tag{6}$$

is related with the solution $P_t(x_1,x_2)$ of the master equation for the ZRP in the domain $x_1\leqslant x_2$ through the relation (4), provided that the former satisfies certain constraint. Indeed, when $(x_2-x_1)\geqslant 2$, the equation for probability $P_t(x_1,x_2)$ for the ZRP coincides with one for noninteracting particles (6). For $x_1=x_2-1=x$ the equation for the ZRP

$$\partial_t P_t(x,x+1) = P_t(x-1,x) + uP_t(x,x) - 2P_t(x,x+1) \tag{7}$$

can be obtained from Eq. (6) if we define

$$P_t(x,x) = \frac{1}{u}P_t^0(x,x), \tag{8}$$

which is consistent with Eq. (4). In the case $x_2=x_1=x$, Eq. (6) contains the term $P_t^0(x,x-1)$ which is beyond the "allowed" region $x_1\leqslant x_2$ and, thus, does not carry any physical content. To restore the correct equation for the ZRP

$$\partial_t P_t(x,x) = P_t(x-1,x) - uP_t(x,x), \tag{9}$$

we can redefine this term to compensate the difference between Eqs. (6) and (9). As a result we get the following constraint on $P_t^0(x_1,x_2)$:

$$P_t^0(x,x-1) = (u-1)P_t^0(x-1,x) - (u-2)P_t^0(x,x). \tag{10}$$

Thus, the solution of the free Eq. (6), which satisfies the constraint (10), gives the solution of the master equation for the ZRP in the domain $x_1\leqslant x_2$. Now, we can use the Bethe ansatz for the eigenfunction of the free equation (6)

$$P_t^0(x_1,x_2) = e^{\lambda t}(A_{1,2}z_1^{-x_1}z_2^{-x_2} + A_{2,1}z_1^{-x_2}z_2^{-x_1}). \tag{11}$$

Its substitution to Eq. (6) results in the expression for the eigenvalue:

$$\lambda = z_1 + z_2 - 2. \tag{12}$$

The ansatz (11) to be consistent with the constraint (10), the amplitudes $A_{1,2}$ and $A_{2,1}$ should satisfy the relation

$$\frac{A_{1,2}}{A_{2,1}} = -\frac{(2-u)-(1-u)z_2-z_1}{(2-u)-(1-u)z_1-z_2}, \tag{13}$$

which together with the cyclic boundary conditions, $P_t^0(x_1,x_2)=P_t^0(x_2,x_1+N)$, results in the system of two algebraic equations. The first one is the following

$$z_1^{-N} = -\frac{(2-u)-(1-u)z_2-z_1}{(2-u)-(1-u)z_1-z_2}, \tag{14}$$

while the second can be obtained by the change $z_1\leftrightarrow z_2$.

B. Many-particle sector

To generalize these results to the case of arbitrary p let us consider the configuration with two neighboring sites $(x-1)$ and x having occupation numbers m and n, respectively. Let us explicitly write down the terms of the master equation corresponding to the transition into and from this configuration due to a particle jump into and from the site x, respectively,

$$\begin{aligned}\partial_t P_t(\ldots,(x-1)^m,(x)^n,\ldots) &= \ldots + u(m+1)P_t(\ldots,(x-1)^{m+1},(x)^{n-1},\ldots)\\ &\quad - u(n)P_t(\ldots,(x-1)^m,(x)^n,\ldots) + \cdots.\end{aligned} \tag{15}$$

Here $(x)^n$ denotes n successive arguments x of the function $P_t(x_1,\ldots,x_p)$. In terms of $P_t^0(x_1,\ldots,x_p)$, which is related to $P_t(x_1,\ldots,x_p)$ according to Eq. (4), this equation looks as follows:

$$\begin{aligned}\partial_t P_t^0(\ldots,(x-1)^m,(x)^n,\ldots) &= \cdots + u(n)\times[P_t^0(\ldots,(x-1)^{m+1},(x)^{n-1},\ldots)\\ &\quad - P_t^0(\ldots,(x-1)^m,(x)^n,\ldots)] + \cdots.\end{aligned} \tag{16}$$

Note that in this form the coefficient before the term in square brackets is equal to $u(n)$, i.e., does not depend on the number m of particles in the site $(x-1)$. Thus, the rhs of the master equation expressed through $P_t^0(x_1,\ldots,x_p)$ is the sum of one-site factors similar to those in Eq. (16) for all nonempty sites. Equating such term with one from the equation for noninteracting particles,

$$\partial_t P_t^0(x_1,\ldots,x_p) = \sum_{i=1}^{p} [P_t^0(\ldots,x_i-1,\ldots) - P_t^0(\ldots,x_i,\ldots)], \tag{17}$$

we obtain the following constraint for $P_t^0(x_1,\ldots,x_p)$,

$$\begin{aligned}&[u(n)-1]P_t^0(\ldots,(x),(x+1)^{n-1},\ldots)\\ &\quad-\sum_{j=2}^{n} P_t^0(\ldots,(x+1)^{j-1},(x),(x+1)^{n-j},\ldots)\\ &\quad-[u(n)-n]P_t^0(\ldots,(x+1)^n,\ldots)=0.\end{aligned} \tag{18}$$

Such relations for all n are to be understood as a redefinition for the terms outside of allowed region $x_1 \leqslant \ldots \leqslant x_p$. The Bethe ansatz to be applicable, the relation (18) should be reducible to the two-particle constraint (10). This can be proved by induction. To this end, we assume that similar relations including $u(k)$ are reducible for all $k<n$. Then starting from the relation (18), which includes rate $u(n)$, we apply Eq. (10) to every pair $(x,x-1)$ of arguments of the function $P_t^0(\ldots)$ under the sum and require the result to be a similar relation for $u(n-1)$. We obtain the following recurrent formula for the rates

$$u(n) = 1-(1-u)u(n-1), \tag{19}$$

which can be solved in terms of q numbers

$$u(n) = [n]_q = \frac{1-q^n}{1-q}, \tag{20}$$

where

$$q = u-1. \tag{21}$$

Further generalization of two-particle results is straightforward. We use the Bethe ansatz for the eigenfunction $P_t^0(x_1,\ldots,x_p)$ of Eq. (17):

$$P_t^0(x_1,\ldots,x_p) = e^{\lambda t}\sum_{\{\sigma_1,\ldots,\sigma_p\}} A_{\{\sigma_1,\ldots,\sigma_p\}}\prod_{i=1}^{p} z_{\sigma_i}^{-x_i}. \tag{22}$$

Here $z_1,\ldots,z_p$ are complex numbers, the summation is taken over all $p!$ permutations $\{\sigma_1,\ldots,\sigma_n\}$ of the numbers $(1,\ldots,p)$, and the coordinates of particles are ordered in the increasing order $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_p$.

Substituting Eq. (22) into Eq. (17) we obtain the expression for the eigenvalue,

$$\lambda = \sum_{i=1}^{p} z_i - p. \tag{23}$$

The numbers $(z_1,\ldots,z_p)$ are to be defined from the Bethe equations,

$$z_i^{-N} = (-1)^{p-1}\prod_{j=1}^{p}\frac{(2-u)-(1-u)z_j-z_i}{(2-u)-(1-u)z_i-z_j}, \tag{24}$$

which follow from the condition of compatibility of cyclic boundary conditions with the constraint (10).

For the sake of convenience in the following discussion we use the parameter q defined in Eq. (21) rather than u. We should note that the appearance of q numbers as the condition of the Bethe ansatz integrability is not unexpected. The notion of q deformation naturally appears in context of Bethe ansatz solvable models characterized by the trigonometric R matrix. In algebraic language this is the consequence of the fundamental Yang-Baxter equation which leads to q-commutation relations between the local operators constituting the transfer matrices [19]. One of such models, q-boson totally asymmetric diffusion model [7], turns out to be directly related to the model we consider. In order to see the correspondence, let us formally write the distribution $P_t(C)$ as a vector of state

$$|P_t(C)\rangle = \sum_{\{C\}} P_t(C)|C\rangle, \tag{25}$$

where C is a configuration of particles on the lattice, $C=\{n_1,\ldots,n_N\}$, and the summation is over all configurations. Consider the algebra generated by the operators $B_j, B_j^+, \mathcal{N}_j$, which act on the occupation number $n_j>0$ of each site j of the lattice as follows:

$$B_j|n_j\rangle = |n_j-1\rangle \tag{26}$$

$$B_j^+|n_j\rangle = [n_j+1]_q|n_j+1\rangle \tag{27}$$

$$\mathcal{N}_j|n_j\rangle = n_j|n_j\rangle. \tag{28}$$

The state $|0\rangle$ plays the role of vacuum state

$$B_j|0_j\rangle = 0. \tag{29}$$

Then, the master equation (3) with the rates given by Eq. (20) can be written in the form of the imaginary time Schrödinger equation

$$\partial_t|P_t(C)\rangle = -\mathbf{H}|P_t(C)\rangle \tag{30}$$

where the amiltonian $\mathbf{H}$ is given in terms of the operators (26) and (27),

$$\mathbf{H} = -\sum_j (B_{j-1}^+B_j - B_j^+B_j). \tag{31}$$

One can directly check that the operators $B_j, B_j^+, \mathcal{N}_j$ satisfy the following commutation relations:

$$[B_j, B_k^+] = q^{\mathcal{N}_j}\delta_{jk}, \tag{32}$$

$$[\mathcal{N}_k, B_j] = -B_j\delta_{jk}, \tag{33}$$

$$[\mathcal{N}_k, B_j^+] = B_j^+\delta_{jk}. \tag{34}$$

These commutation relations and the Hamiltonian (31) give us, up to the change of notations $q \to q^{-2}$, the definition of the

q-boson totally asymmetric diffusion model. Obviously, the dynamical rules of the ZRP with the rates given by Eq. (20) is nothing but the explicit realization of this hamiltonian. Its integrability has been shown in Ref. [7] via algebraic Bethe ansatz and two-particle diffusion on the infinite lattice has been considered. Note that while in Ref. [7] the q-boson totally asymmetric diffusion model was initially defined in terms of q-boson operators, we started from the ZRP with arbitrary rates and came to q numbers as the integrability condition.

Let us first take a qualitative look at the behavior of the ZRP resulted by the choice (20) of the rates $u(n)$ for different values of q. In the limit $q\to 1$ the q-numbers degenerate into simple numbers. Therefore, the rates are given by $u(n)=n$, which corresponds to the diffusion of noninteracting particles. The Bethe equations in this case decouple to the form $z_j^N=1$ as is expected in noninteracting case. In the domain $q>1$ the rates $u(n)$ grow exponentially with n, which corresponds to the interaction between particles effectively accelerating free diffusive motion, i.e., the higher the density of particles the faster is their mean velocity. In the limit $q\to\infty$ the model is equivalent to $n=1$ drop-push model [8], which is confirmed by the same form of Bethe equations [9]. In the domain $0<q<1$ the rates $u(n)$ grow monotonously from $(q+1)$ for $n=2$ to $1/(1-q)$ for $n\to\infty$, resulting in the interaction between particles slowing down the particle flow compared to the one of noninteracting diffusing particles. When $q=0$, all the rates do not depend on the number of particles, i.e., $u(n)=1$. This case (also referred to as phase model [20]), can be mapped on the totally asymmetric ASEP by insertion of one extra bond before every particle. At last, in the domain $-1<q<0$ the rates $u(n)$ also saturate to the constant $1/(1-q)$ with growth of n, though oscillating around this value. As it has been mentioned above, the ZRP served as an example of the nonequilibrium system with the condensation transition. In our case, however, the condensation is absent as the rates $u(n)$ defined above do not satisfy Evans criteria according to which the condensation in the ZRP occurs if the rates saturate to a constant slower than $2/n$.

It is interesting that the recursion relation (19) rewritten in terms of the quantity $[1-u(n)]$ coincide with one for the toppling probabilities μ_n, imposed by the requirement of the Bethe ansatz integrability in the ASAP. In fact, the ASAP can be represented as a special case of the discrete time ZRP viewed from the reference frame moving together with an avalanche. This situation, however, is quite different from one considered here. In the moving reference frame all particles hop definitely to the next site except one from an active site, which is the only site with multiple occupation. In that case quantity $1-\mu_n$ plays the role of probability of hopping of a particle out of this site. This dynamics leads to the picture similar to the ZRP on an inhomogeneous lattice with one attractive site. Such ZRP was shown to exhibit the condensation transition. In terms of the avalanche processes it is the transition from the intermittent to continuous flow. Despite the nonuniform stationary state of the ASAP in discrete time [17], its Bethe ansatz solution was based on the continuous time picture considered on the ensemble of equiprobable configurations with at most one-particle occupation. The avalanches were accounted in the rates of transitions between these configurations, generating infinite series in the master equation.

To use the eigenfunctions obtained for the construction of particular solutions one should first question if they form complete orthogonal basis. In general this question is not easy to answer, as the set of solutions of the Bethe equations is not known. Some arguments have been given [3,21] for the the totally asymmetric exclusion process due to its connection with the asymmetric six-vertex model [22]. The long time characteristics of the process can, however, be analyzed without discussing this question. To this end, we can use the properties of the largest eigenvalue for slightly modified equation, which describes the generating function of total distance traveled by particles [23]. The advantages of this approach are first that the uniqueness of the largest eigenvalue is guaranteed by Perron-Frobenius theorem. Second, corresponding solution of the Bethe equations can be easily identified as it corresponds to the stationary state of the process. To give an example of application of the above results we perform this analysis in Sec. IV.

III. STRUCTURE OF THE STATIONARY STATE

Before going to the results of the analysis of the Bethe equations, let us first look at the structure of the stationary measure of the model. It is characterized by the partition function

$$Z(N,p)=\sum_{\{n_1,\ldots,n_p=1\}}^{\infty}\delta(n_1+\cdots+n_N-p)\prod_{i=1}^{N}f(n_i), \quad (35)$$

where one-site weight $f(n)=1/[n]_q!$, defined in Eq. (2), is expressed through the q factorial $[n]_q!=\prod_{k=1}^{n}[k]_q$. In the limit $q\to 1$, q factorial turns into simple factorial, as it should be in the noninteracting case. The sum in Eq. (35) can be presented in the form of the contour integral

$$Z(N,p)=\frac{1}{2\pi i}\oint\frac{(F(z))^N}{z^{p+1}}dz, \quad (36)$$

where the series $F(z)=\sum_{n=0}^{\infty}f(n)z^n$ can be summed to the infinite product due to the q-binomial theorem [24]: for $|q|<1$

$$F(z)=\sum_{n=0}^{\infty}\frac{(z(1-q))^n}{(q;q)_n}=(z(1-q);q)_{\infty}^{-1} \quad (37)$$

and for $|q|>1$

$$F(z)=\sum_{n=0}^{\infty}\frac{(z(1-q^{-1}))^n q^{-n(n-1)/2}}{(q^{-1};q^{-1})_n}=(z(q^{-1}-1);q^{-1})_{\infty}. \quad (38)$$

Here, we used the notation $(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$ for shifted q factorial. The above q series are known as q analogs of the exponential function e^z, which can be restored in the limit $q\to 1$. The presence of q analogs of the functions, which appear in the case of noninteracting particles, is a direct consequence of the replacement of simple numbers by q numbers in the expression of the rates of hopping. In the thermodynamic limit, $N\to\infty$, $p\to\infty$, $p/N=\rho$, the integral (36) can be calculated in the saddle point approximation. The equation for the saddle point z_0,

$$\rho = z_0 \log' F(z_0), \tag{39}$$

contains the logarithmic derivative of $F(z)$, which can be evaluated using the product form of $F(z)$ (37) and (38). As a result we obtain the following relations between z_0, ρ, and q:

$$\rho = z_0(1-q)\sum_{n=0}^{\infty} \frac{q^n}{1 - z_0(1-q)q^n}, \tag{40}$$

$$\rho = z_0(1-q^{-1})\sum_{n=0}^{\infty} \frac{q^{-n}}{1 - z_0(q^{-1}-1)q^{-n}} \tag{41}$$

for $|q|<1$ and $|q|>1$, respectively. Below the same equations will appear in a different context from the analysis of Bethe ansatz equations. Then, the partition function,

$$Z(N,p) = \frac{(F(z_0))^N}{z_0^p}, \tag{42}$$

can be used to calculate stationary state correlations such as the speed, $v=\langle u(n)\rangle$, i.e., the average hopping rate out of a site

$$v = \frac{Z(N,p-1)}{Z(N,p)} = z_0 \tag{43}$$

or the probability distribution of the number of particles in a site

$$P(n) = f(n)\frac{Z(N-1,p-n)}{Z(N,p)} = \frac{1}{[n]_q!}\frac{z_0^n}{F(z_0)}. \tag{44}$$

IV. THE LONG TIME BEHAVIOR FROM THE BETHE EQUATIONS

To obtain any results beyond the stationary correlations one needs to analyze the above Bethe ansatz solution. Since similar analysis has been done several times before [21,25–27], we do not give the detailed calculations here. Instead we outline the main points of the solution to emphasize the connection with the formulas obtained from the analysis of stationary measure.

Consider the generating function $\mathcal{F}_t(C)=\sum_{Y=0}^{\infty} P_t(C,Y)e^{\gamma Y}$, where $P_t(C,Y)$ is the joint probability for the system to be in the configuration C at time t and the total distance traveled by particles being Y. The only difference of the equation for $\mathcal{F}_t(C)$ from Eq. (3) is the coefficient e^{γ} before the first term under the sum in the rhs, which corresponds to the increasing of traveled distance by unity due to the hopping of one particle. At large time, $t\to\infty$, the behavior of the generating function of the distance Y_t traveled by particles up to time t, $\langle e^{\gamma Y_t}\rangle = \sum_C \mathcal{F}_t(C)$, is determined by the largest eigenvalue of the equation for $\mathcal{F}_t(C)$:

$$\lambda_0(\gamma) = \lim_{t\to\infty} \frac{\ln\langle e^{\gamma Y_t}\rangle}{t}. \tag{45}$$

Using the ansatz (4) and (22) for the eigenfunction of the equation for $\mathcal{F}_t(C)$ we can repeat all the above arguments. Then, if we make the variable change $x_i = 1-e^{\gamma}z_i$, the eigenvalue and the Bethe equations will simplify to the following form:

$$\lambda(\gamma) = -\sum_{i=1}^{p} x_i, \tag{46}$$

$$e^{\gamma N}(1-x_i)^{-N} = (-1)^{p-1}\prod_{j=1}^{p} \frac{x_i - qx_j}{x_j - qx_i}. \tag{47}$$

In these variables the rhs of the Bethe equations coincides with one for the partially asymmetric ASEP and the ASAP. This allows us to modify the techniques developed for the analysis of these processes.

In the thermodynamic limit, $N\to\infty$, $p\to\infty$, $p/N=\rho$, we assume that the roots of the Bethe equations (47) are distributed in the complex plain along some continuous contour Γ with the analytical density $R(x)$, so that the sum of values of a function $f(x)$ at the roots is given by

$$\sum_{i=1}^{p} f(x_i) = N\int_{\Gamma} f(x)R(x)dx. \tag{48}$$

After taking the logarithm and replacing the sum by the integral, the system of Eq. (47) can be reformulated in terms of single integral equation for the density. The particular solution corresponding to the largest eigenvalue is specified by the appropriate choice of branch of the logarithm. Then the integral equation should be solved for a particular form of the contour, which is not known as *a priori*, and being first assumed should be self-consistently checked after the solution has been obtained. In practice, however, analytical solution is possible in the very limited number of cases. Particularly, one, corresponding to the contour closed around zero, yields the density

$$R^{(0)}(x) = \frac{1}{2\pi i x}\left(\rho - \sum_{n=1}^{\infty} \frac{x^n}{1-q^n}\right) \tag{49}$$

for $|q|<1$ and

$$R^{(0)}(x) = \frac{1}{2\pi i x}\left(\rho + \sum_{n=1}^{\infty} \frac{x^n}{1-q^{-n}}\right) \tag{50}$$

for $|q|>1$. This case corresponds to $\gamma=0$ and hence $\lambda(\gamma)=0$. Since $R^{(0)}(x)$ is analytic in the ring $0<|x|<1$, the integration of it along any contour closed in this ring does not depend on its form. Therefore, to fix the form of the contour additional constraints are necessary. Such constraint appears if we require that the density preserves its analyticity when γ deviates from zero and the contour becomes discontinuous. This constraint implies that the density $R^{(0)}(x)$ vanishes at the break point x_c, which is a crossing point of the contour Γ and the real axis:

$$R^{(0)}(x_c) = 0. \tag{51}$$

This equation was first obtained by Bukman and Shore as a conical point condition for the asymmetric six-vertex model [25]. It is remarkable that after the resummation of series in Eqs. (49) and (50), Eq. (51) coincides with Eqs. (40) and (41) up to the replacements

$$x_c = z_0(1-q) \quad \text{and} \quad x_c = z_0(q^{-1}-1) \tag{52}$$

for $|q|<1$ and $|q|>1$, respectively. Remember that z_0 was shown to coincide with the speed v. It will be clear below that the same relation between $v=(\lambda_0'|_{\gamma=0})/N$ and x_c follows directly from the expression for $\lambda_0(\gamma)$ obtained from the Bethe equations, without appealing to the partition function.

Further analysis is related with the calculation of finite size corrections to the above expression of $R^{(0)}(x)$, which makes possible to probe into the nonzero values of γ. This can be done with the help of method developed in Refs. [21,23]. Its essential part is the construction of the inverse expansion $Z^{-1}(x)$ to the function of the number of roots $Z(x)=\int^x R(x)dx$ near the break point x_c. Since the derivative of $Z(x)$ in the thermodynamic limit vanishes [see Eq. (51)], its inverse expansion reveals the square root singularity, which in its turn becomes the origin of $1/\sqrt{N}$ terms in the finite size expansion of $R(x)$. As a result we obtain the following parametric dependence of $R(x)$ on γ, both being represented as functions of the same parameter C:

$$R_s = R_s^{(0)} - \frac{1}{N^{3/2}}\frac{1}{2\pi i}\frac{q^{|s|}}{1-q^{|s|}}\sum_{n=0}^{\infty}\left(\frac{i}{2N}\right)^n \times \frac{\Gamma\left(n+\frac{3}{2}\right)}{\pi^{n+\frac{3}{2}}}\frac{c_{2n+1,s}}{\sqrt{2i}}\mathrm{Li}_{n+\frac{3}{2}}(C), \tag{53}$$

$$\gamma = \frac{1}{N^{3/2}}\sum_{n=0}^{\infty}\left(\frac{i}{2N}\right)^n \frac{\Gamma\left(n+\frac{3}{2}\right)}{\pi^{n+\frac{3}{2}}}\frac{\overline{c_{2n+1}}}{\sqrt{2i}}\mathrm{Li}_{n+\frac{3}{2}}(C). \tag{54}$$

Here R_s and $R_s^{(0)}$ are the Loraunt coefficients of $R(x)$ and $R^{(0)}(x)$, respectively, defined as follows:

$$R(x) = \sum_{s=-\infty}^{\infty} R_s/x^{s+1}, \tag{55}$$

$c_{2n+1,s}$ and $\overline{c_{2n+1}}$ are the coefficients of x^n in $(\sum_{k=0}^{\infty}a_k x^k)^s$ and $\ln(\sum_{k=0}^{\infty}a_k x^k)$, respectively, where a_n are the coefficients of the inverse expansion $Z^{-1}(x)$ near the point x_c. The location of x_c is to be self-consistently defined from the equation $R(x_c)=0$. For the first three orders of $1/\sqrt{N}$ expansion the coefficients a_n can be obtained from the inverse expansion of zero order function $Z^{(0)}(x)=\int^x R^{(0)}(x)dx$, while $R^{(0)}(x)$ has been obtained above. To evaluate the sum over the roots, one needs to integrate along the contour Γ, which can be obtained from the initial closed contour by cutting out small segment connecting two roots closest to the point x_c. For a function which is defined by the expansion

$$f(x) = \sum_{s=1}^{\infty} f_s x^s, \tag{56}$$

this yields

$$\sum_{i=1}^{p} f(x_i) = 2\pi i N \sum_{s=1}^{\infty} q^{\pm s} f_s R_s, \tag{57}$$

minus and plus in power of q being for $|q|<1$ and $|q|>1$, respectively. Finally the point x_c enters all the results through the coefficients $c_{2n+1,s}$ and $\overline{c_{2n+1}}$. It is related with the physical quantities through Eqs. (40), (41), (43), and (52). We use this relation to write the final results as a function of speed v, density ρ, and the parameter q. Below we give the expression for the largest eigenvalue in the scaling limit $\gamma N^{3/2}=\text{const}, N\to\infty$:

$$\lambda_0(\gamma) = Nv\gamma + k_1 G(k_2\gamma). \tag{58}$$

Here the function $G(x)$ has the following parametric form:

$$G(x) = -\mathrm{Li}_{5/2}(C), \tag{59}$$

$$x = -\mathrm{Li}_{3/2}(C) \tag{60}$$

with $\mathrm{Li}_\alpha(x)=\sum_{k=1}^{\infty}x^k/k^\alpha$ is the function of polylogarithm and the constants k_1, k_2 are

$$k_1 = \frac{1}{N^{3/2}}\sqrt{\frac{v/(1-q)}{8\pi}}\,\frac{g_q''(v(1-q))}{[g_q'(v(1-q))]^{5/2}}, \tag{61}$$

$$k_2 = N^{3/2}\sqrt{2\pi v(1-q)g_q'(v(1-q))} \tag{62}$$

for $|q|<1$, and

$$k_1 = \frac{1}{N^{3/2}}\sqrt{\frac{v/(1-q^{-1})}{8\pi}}\,\frac{g_{1/q}''(v(q^{-1}-1))}{[g_{1/q}'(v(q^{-1}-1))]^{5/2}}, \tag{63}$$

$$k_2 = -N^{3/2}\sqrt{2\pi v(1-q^{-1})g_{1/q}'(v(q^{-1}-1))} \tag{64}$$

for $|q|>1$, where

$$g_\nu(x) = \sum_{n=1}^{\infty}\frac{x^n}{1-\nu^n}. \tag{65}$$

The scaling form of the function $G(x)$ was suggested to be universal for Kardar-Parisi-Zhang (KPZ) universality class [28,29]. Using the generating function obtained one can evaluate all the cumulants of the traveled distance

$$\lim_{t\to\infty}\frac{\langle Y_t^n\rangle_c}{t} = \left.\frac{\partial^n\lambda_0(\gamma)}{\partial\gamma^n}\right|_{\gamma=0}. \tag{66}$$

The large deviation function, $\mathrm{ldf}(x)=\lim_{t\to\infty}\ln P(Y_t=x)/t$, can be also obtained as a Legendre transformation of $\lambda_0(\gamma)$.

V. SUMMARY AND DISCUSSION

To summarize, we apply the Bethe ansatz to solve zero-range process with the totally asymmetric dynamics on a ring. The eigenfunctions of the master equation have the form of the Bethe function weighted with the stationary weights of corresponding particle configurations. The requirement of Bethe ansatz integrability leads to the special choice of the rates of hopping of particle out of a site. The rates should be q numbers, $[n]_q$, generalizing the case of noninteracting diffusing particles, where the rate is equal to n, the number of particles at a site of departure. The noninteracting case can be restored in the limit $q \to 1$. Two other limiting cases, $q=0$ and $q \to \infty$, reproduce well known totally asymmetric exclusion process and drop-push model, respectively. The case of general q is shown to be equivalent to the q-boson totally asymmetric diffusion model. Continuing analogy with noninteracting case, we show that many quantities characterizing the stationary state correlations of the model turn out q analogs of corresponding functions appearing in the noninteracting case. To provide an example of application of the Bethe ansatz solution obtained, we derive the expression for the large time limit of the generating function of cumulants of the total distance traveled by particles. It has a universal form specific for KPZ universality class. The question whether the q-boson totally asymmetric diffusion model belongs to KPZ class was addressed in concluding remarks in Ref. [7]. The result (58)–(60) is an argument in favor of this assumption.

In connection with above results the following questions appear. First, is it possible to generalize the proposed combination of the Bethe ansatz with stationary weights to any other processes with nonuniform stationary state, say asymmetric exclusion process with parallel update? The consideration of the associated vertex models is also of interest. The different weights of vertex configurations depending on the order of vertices would result in the appearance of nonlocal interaction. Second, can one apply the matrix product method to study the exclusion process with long range interaction associated with zero-range process considered here to probe into spatially inhomogeneous situation, e.g., at the open chain. Appearance of q numbers seems to be an indication of this possibility. We expect that the matrix product ansatz should be again appropriately weighted with stationary weights of some homogeneous system. The consideration of such a system is attractive, as the extra parameter q could result in a reacher phase diagram compared to the usual totally asymmetric exclusion process. Third, it is interesting to establish correspondence of the large scale behavior of the proposed process with KPZ equation.

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